# Splitting Methods for Dry Frictional Contact Problems in Rigid Multibody Systems: Preliminary Performance Results

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## Abstract

A splitting method for solving LCP based models of dry frictional contact problems in rigid multibody systems based on box MLCP solver is presented. Since such methods rely on fast and robust box MLCP solvers, several methods are reviewed and their performance is compared both on random problems and on simulation data. We provide data illustrating the convergence rate of the splitting method which demonstrates that they present a viable alternative to currently available methods.

**CR Categories:** G.1.6 [Mathematics of Computing]: Optimization—Nonlinear Programming I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling—Physically based modeling I.3.7 [Computer Graphics]: Three-Dimensional Graphics and Realism—Virtual Reality I.6.8 [Simulation and Modeling]: Types of Simulation—Animation

**Keywords:** physics based modeling, constraints, LCP, dry friction, rigid multibody dynamics

## 1 Introduction

Rigid multibody dynamics with frictional contacts is a well established topic in interactive graphics and in the engineering literature. There are several papers on the topics each year in the SIGGRAPH proceedings since the late 80s. The application domain in the context of interactive 3D graphics covers operator training for ground vehicles, robotics, gaming, and animation authoring tools, especially for movies.

Several software packages are available on the market but few can simulate rigid multibody systems with dry frictional contacts satisfactorily, though many perform well on problems without dry friction. The issues range from poor stability, lack of robustness in the case of degenerate or ill-posed problems, poor scalability, to anomalous friction forces. Given that a consistent and solvable mathematical model for dry frictional contacts was not published until 1997[Anitescu and Potra 1997], this fact is not very surprising.

Alternative solution methods have been suggested for solving dry frictional contact problems including spring and damper systems, pairwise impulse models[Mirtich and Canny 1995], and many variants based on Linear or Nonlinear Complementarity Problems (LCPs and NLCPs respectively). For the latter, we cite[Al-Fahed et al. 1991][Pang and Trinkle 1996] [Stewart and Trinkle 1996][Anitescu and Potra 1997][Anitescu et al. 1999]. Spring-damper models will not be discussed as they tend to be unstable. Pairwise impulse based models amount to Gauss-Seidel iterative processes and we will provide data on such methods below.

LCP models are split in acceleration based models [Al-Fahed et al. 1991][Trinkle et al. 1997][Pang and Trinkle 1996][Pfeiffer and Glocker 1996][Tzitzouris 2001], and velocity time stepping models [Stewart 1997][Stewart and Trinkle 1996][Anitescu et al. 1999][Anitescu and Potra 1997]. The first type is not guaranteed to be solvable except for very small friction coefficients[Pang and Trinkle 1996]. There are discontinuities inherent in dry friction at transitions from static to kinetic friction as explained in[Stewart and Trinkle 1996]. These lead to impulses i.e., infinite instantaneous accelerations, even when there is no collision. Therefore, a model which requires the computation of accelerations is fundamentally flawed, unless all impulses are detected and processed appropriately. Integrating over the accelerations, it is possible to obtain a solvable velocity time stepping model. This is the basis of the present work.

There is no general solution method which is guaranteed to work for any given LCP, except for total enumeration has complexity of $O(2^n)$ for problems of size $n$. The most robust method is still that of Lemke[Lemke 1965][Sargent 1978] [Júdice et al. 1992]. Only this method can compute a solution of the solvable dry frictional contact models[Stewart and Trinkle 1996][Anitescu and Potra 1997]. However, Lemke's algorithm only solves these problems after reducing the original mixed linear complementarity problem (MLCP) to an LCP with Schur complements. This is expensive and cannot work on degenerate or ill-conditioned problems. Additionally, Lemke's algorithm cannot reuse a previous solution as an advanced starting point which leads to considerable inefficiency, especially near equilibrium.

Computing dry frictional contact forces is hard because of the coupling between the tangential (friction) and normal components. In the Coulomb model for instance, a contact point is in static friction, with zero tangential contact velocity, until the magnitude of the friction force reaches the product of the friction coefficient with the normal force. Because of this coupling, the problem is no longer a quadratic program (QP) but instead, it is an LCP which does not correspond to a minimization or saddle point problem. However, given the normal contact forces at a given point, finding the tangential friction forces corresponds to solving a QP and alternately, given the tangential friction forces at a contact point, finding the normal contact forces corresponds to solving another QP. This has been used [Šimunović and Saigal 1994][Dostál et al. 2002] to construct operator splitting methods which iteratively estimate the two components. Since box QP solvers can use advanced starting points, it is possible to quickly solve a sequence of QPs which, hopefully, converges to a correct solution of the more accurate LCP model. Operator splitting has already been used for simulating rigid multibody systems; a two pass method was used in[Milenkovic and Schmidl 2001] for instance. Splitting methods are also used in commercial software libraries though the evidence for that is circumstantial. The convergence rate of operator splitting doesn't seem to have been investigated and as we show below, two pass methods do not seem to yield very accurate results and is

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sensitive on the starting point.

A performance and robustness review of existing box QP solvers has not been published yet, especially with regards to degenerate contacts. In the rigid body case, a single box resting on a plane can lead to a degenerate problem and therefore, a solver which is robust against degeneracy and ill-posed problems is badly needed. The current paper offers such a review.

In what follows, we will concentrate on numerical experiments which demonstrate that good convergence can be obtained in practice, leaving the proof for future work. We will review the performance of solvers for LCPs and QPs with box constraints on both random problems and real data sets extracted from simulations.

## 2 Linear Complementarity

The LCP is defined for a square, $n \times n$, matrix $H$ and an $n$ dimensional vector $q$ as the problem of finding $n$ dimensional solution vectors $z$ and $w$ such that:

$$0 \leq z \perp Hz + q = w \geq 0, \tag{1}$$

where the perpendicularity is understood component wise i.e., $z_i, w_i \geq 0, z_i w_i = 0$. The vector $w$ is sometimes called the "slack" variable. Extensive coverage of this is found in [Murty 1988] and[Cottle et al. 1992].

A simple extension of the LCP is to impose lower and upper bounds on the solution vector $z$, $l$ and $u$, respectively, which may be finite or infinite. The residual vector, $w$, is now split into positive and negative components thus: $w = w_+ - w_-$ with $w_+, w_- \geq 0, w_+ w_- = 0$. This leads to one form of the Mixed LCP which we called the box LCP. The definition is as follows:

$$\begin{aligned} Hz + q = w_+ - w_-, \\ 0 \leq z - l \perp w_+ \geq 0, \quad 0 \leq u - z \perp w_- \geq 0. \end{aligned} \tag{2}$$

A review of many algorithms for solving LCPs is found in [Júdice 1994].

## 3 Multibody Dynamics

We consider a multibody system with generalized coordinates $q \in \mathbb{R}^n$, generalized velocities $v = \dot{q}$, and mass matrix $M(q) \in \mathbb{R}^{n^2}$. The generalized forces acting on the system are $F \in \mathbb{R}^n$ (including the non-inertial forces). The system is subject to holonomic constraints, $\Phi(q,t) = 0$, and inequality constraints, $\Xi(q,t) \geq 0$. The Jacobians for these are written as $G$ and $N$, respectively, and the Lagrange multiplier associated with them are written $\lambda$ and $\nu$, resp. Velocity constraints are not covered to simplify the notation. Detailed description of constrained dynamical systems are found in [Goldstein 1980][Layton 1998]. A more detailed description of the notation used here for multibody systems is found in [Anitescu and Potra 1997]. We are assuming descriptor form here but mixed representations are possible[Trinkle et al. 1997].

Using this notation, we have the following equations of motion for a multibody system with frictionless contacts:

$$\begin{aligned} M\dot{v} - G^t \lambda - N^t \nu - F = 0 \\ \Phi(q,t) = 0 \\ 0 \leq \nu \perp \Xi(q,t) \geq 0. \end{aligned} \tag{3}$$

This system in Eq. (3) is a form of differential algebraic equation (DAE) (see ref. [Hairer and Wanner 1996] for instance). However, due to the inequality constraints, this is not a standard problem. In[Stewart and Trinkle 1996], it is shown that these systems are differential inclusions. We shall concentrate on the solution methods for the static problem, common to all numerical integration procedures.

Numerical integration of Eq. 3 requires solving the following linear system:

$$\begin{bmatrix} M & -G^t & -N^t \\ G & 0 & 0 \\ N & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \lambda \\ \nu \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \delta \end{bmatrix}, \quad 0 \leq \nu \perp \delta \geq 0. \tag{4}$$

The vectors $a, b, c$ depend on the discretization details. The system of equations in Eq. 4 form a mixed linear complementarity model (MLCP).

## 4 Dry Frictional Contact Models

For a pair of bodies with indicies $i_1, i_2$ which are in close proximity, it is assumed there is at least one signed distance functions: $\xi^{(j)}(q^{(i_1)}, q^{(i_2)})$ such that $\xi^{(j)} > 0$ if the bodies are separated, $\xi^{(j)} = 0$ if they are touching and $\xi^{(j)} < 0$ if they are penetrating. This function $\xi^{(j)}$ represents a potential point of contact. In practice, several of these signed distance functions are used for a given pair of rigid bodies. To compute the normal force required to prevent interpenetration, we impose the kinematic constraint $\xi^{(j)} \geq 0$ with a Lagrange multiplier $\nu^{(j)} \geq 0$. Contacts are non adhesive and therefore, $\nu^{(j)} \geq 0$ and is complementary to $\xi^{(j)} \geq 0$.

The gradient of $\xi^{(j)}$ defines a vector $\bar{n}^{(j)}$ normal to the tangent contact plane. This is spanned by a set of basis vectors $\{d^{(j,1)}, d^{(j,2)}, \ldots, d^{(j,n_d)}\}$. The simplest case, we use just two orthogonal vectors but some models require the use of a large number of non-orthogonal vectors to reduce anisotropy[Anitescu and Potra 1997]. Using these, we can compute the projection $D^{(j)}$ such that the relative tangential velocity in the contact plane is given by $\bar{v}^{(j)} = D^{(j)} v$ and the tangential contact force is given by $D^{(j)T} \beta^{(j)}$.

Static friction corresponds to zero contact velocity i.e., $D^{(j)} v = 0$. For kinematic friction, the magnitude of the tangential contact force is given by the product of the kinetic friction coefficient and the magnitude of the normal force at the contact point, $\|\beta^{(j)}\| = \mu_k^{(j)} \nu^{(j)}$, and the direction is anti-parallel to the tangential contact velocity. We only cover the case of isotropic friction here and we use only one friction coefficient for both cases. Two simplifications are possible.

First, one can linearize the norm operator. A vector in the tangent plane can be written as: $x^{(j)} = \sum_k \alpha_k d^{(j,k)}$ with $\alpha_k \geq 0$, with only few non-zero $\alpha_k$. We introduce $E^{(i)} = (1, 1, \ldots, 1)^T)$ such that $\|x^{(j)}\| \simeq E^{(j)^T} x^{(j)} = \sum_k \alpha_k$. A linearized approximation of the Coulomb friction model consists of the following set of complementarity conditions:

$$\begin{aligned} 0 \leq D^{(j)} v + E^{(j)} \sigma^{(j)} \perp \beta^{(j)} \geq 0 \\ 0 \leq \mu^{(j)} \nu^{(j)} - E^{(j)T} \beta^{(j)} \perp \sigma^{(j)} \geq 0. \end{aligned} \tag{5}$$

The second equation tells us that as long as the magnitude of the friction force $|\beta^{(j)}|$ is less than the product of the friction coefficient and the normal force, the sliding velocity $\sigma^{(j)}$ vanishes and the first equation imposes that constraint as $D^{(j)} v = 0$. When the magnitude of the tangential force reaches the maximum, we get a non-zero sliding velocity and the first block of complementarity conditions picks a direction for the tangential force which is nearly anti-parallel to the sliding.

This model[Stewart and Trinkle 1996][Anitescu and Potra

1997], leads to the following MLCP:

$$\begin{bmatrix} M & -G^T & -N^T & -D^T & 0 \\ G & 0 & 0 & 0 & 0 \\ N & 0 & 0 & 0 & 0 \\ D & 0 & 0 & 0 & E \\ 0 & 0 & U & -E^T & 0 \end{bmatrix} \begin{bmatrix} v \\ \lambda \\ \nu \\ \beta \\ \sigma \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho \\ \delta \\ \eta \end{bmatrix}, \tag{6}$$
$$0 \le \begin{bmatrix} \nu^t & \beta^t & \sigma^t \end{bmatrix}^t \perp \begin{bmatrix} \rho^t & \delta^t & \eta^t \end{bmatrix}^t \ge 0,$$

where $a, b, c, d, e$ depend on the discretization of choice, the matrix $U$ is the agglomeration of the friction coefficients, $U = \text{diag}(\mu^{(1)}, \mu^{(2)}, \ldots \mu^{(n_c)})$ for $n_c$ contact points, and $E$ is the agglomeration of the linearized norm operator: $E = \text{diag}(E^{(1)}, E^{(2)}, \ldots E^{(n_c)})$.

This MLCP can be solved using Lemke's method but in order to do this, we need to take two Schur complements. Details for this are found in [Anitescu et al. 1999]. Computing this matrix is far from trivial in terms of numerical work. However, though the Lemke algorithm can process the reduced problem, no extension is guaranteed to solve the original MLCP of Eq. (6).

The second option is to impose a fixed bound on the tangential forces based on an estimate of the normal force. Using two perpendicular directions, $d^{(j,1)}, d^{(j,2)}$, we impose the box bounds $-\mu^{(j)}\bar{\nu}^{(j)} \le \beta^{(j,i)} \le \mu^{(j)}\bar{\nu}^{(j)}$ for $i = 1, 2$, where $\bar{\nu}^{(j)}$ is an approximation of the expected normal force for the given contact point. The constraint equations expressing this model are as follows:

$$\begin{aligned} Dv - \bar{v}_+ + \bar{v}_- &= 0, \\ 0 \le \beta - \underline{\beta} \perp \bar{v}_+ \ge 0, \quad 0 \le &\overline{\beta} - \beta \perp \bar{v}_- \ge 0 \end{aligned} \tag{7}$$

where $\overline{\beta}, \underline{\beta}$ are the lower and upper bounds of $\beta$ respectively, and $\bar{v}_+$ and $\bar{v}_-$ are the positive and negative components of the tangential contact velocity respectively. This model is dissipative and it has most of the properties of the Coulomb model. However, it exhibits anisotropy and has the wrong transition point if the estimate is too far. This can be turned into an iterative scheme though, which can converge to the correct answer. This is a form of operator splitting which requires fast solution of box constrained MCLPs and we provide performance data on this below.

The box friction model leads to the MLCP:

$$\begin{bmatrix} M & -G^T & -N^T & -D^T \\ G & 0 & 0 & 0 \\ N & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \lambda \\ \nu \\ \beta \end{bmatrix} + \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{d} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{\rho} \\ \tilde{\delta} \end{bmatrix}, \tag{8}$$
$$0 \le \nu \perp \tilde{\rho} \ge 0, \quad 0 \le \beta - \underline{\beta} \perp \tilde{\delta} \ge 0, \quad 0 \le \overline{\beta} - \beta \perp \tilde{\delta} \le 0.$$

If the estimates for $\underline{\beta}$ and $\overline{\beta}$ are accurate and if the directions $d^{(j,k)}$ are well-chosen, a solution of the box MLCP will also solve the MLCP of Eq. (6).

# 5 Solvers for Complementarity Problems

There are three types of LCP solvers: pivoting methods, Newton methods, and iterative methods. An extensive review of these algorithms is available in[Júdice 1994]. Except for special cases, LCPs are NP hard problems. A statistical study of solver performance is therefore appropriate.

The pivoting methods include Lemke's almost complementary pivot method (see [Murty 1988], ch. 2), Cottle-Datzig's principal pivot method[Cottle 1968], Keller's principal pivot method[Keller 1973], and Murty's principal pivot method[Murty 1974] among many others. The amount of work done per iteration amounts to a Gauss-Jordan pivot operation[Golub and Van Loan 1996] which is of $O(n^2)$ where $n$ is the size of the problem.

Lemke's method solves the largest class of problems namely, those defined with copositive plus matrices i.e., matrices $H$ for which $y \in \mathbb{R}^n, y \ge 0 \leftarrow Hy \ge 0$ and such that for $y \ge\in \mathbb{R}^n$, if $y^T My = 0$ then, $(H + H^T)y = 0$. This class includes positive semidefinite matrices. Keller's and Cottle-Dantzig's principal pivot methods are guaranteed to work on $P_0$ matrices which are those for which all principal minors are non-negative. This class also includes positive semi-definite matrices. Murty's method only works on $P$ matrices, those for which all principal minors are positive which includes positive definite matrices. With the exception of Murty's principal pivot methods, none of the pivot methods can be started at an advanced point. As we show below, both Lemke's and Keller's method exhibit good performance, performing roughly $n$ pivot operations on average, at least on the class of problems we tested. Murty's and Cottle-Dantzig's methods seem to perform erratically, executing many times more than $n$ pivot operations on problems of size $n$. All these methods can be extended to cover MLCP with box constraints.

Newton's method can be applied to MLCPs[Kelley 1995] and in particular, we have used[Zhang and Gao 2003][Li and Fukushima 2000]. A block pivot method[Kostreva 1978][Júdice and Pires 1994] can be shown to be equivalent to a Newton method without smoothing and without line search. This has been used extensively. The methods with smoothing and line search are more recent and haven't been used extensively yet but are presumed to be more robust but are more complicated to tune with roughly a dozen free parameters. Newton methods require solving a linear system of size $n$ at each step and therefore, each iteration has approximate cost $O(n^3)$. All Newton-type methods can start from an advanced point i.e., a point which is hoped to be near the solution. All Newton methods can solve $P_0$ problems but work better on $P$ problems.

For iterative methods, the matrix $H$ is decomposed as $H = D + L + U$ where $D$ is a block diagonal matrix, $L$ is a strictly lower triangular matrix and $U$ is a strictly upper triangular matrix. We then solve LCPs corresponding to one block of equations, keeping the other ones fixed:

$$0 \le D_{jj} z_j^{(k+1)} + q_j + L_{jj'} z_{j'}^{(k+1)} + U_{jj'} z_{j'}^{(k)} \perp z_j^{(k+1)} \ge 0. \tag{9}$$

Here, $j$ is the set of indicies corresponding to a block and $j'$ is the set of all other indicies. This is a block Gauss-Seidel scheme.

The Gauss-Seidel method is very attractive for its simplicity. However, the convergence rate is $\rho^m$ where $\rho$ is the condition number of the matrix $D^{-1}(L+U)$. Though $\rho < 1$ when $H$ positive definite, it is often very near 1. When working on random problems, condition numbers exceeding $1E7$ made the method unusably slow. On simulation data extracted from a piling problem, the method stagnated for thousands of iterations.

All pairwise methods such as[Mirtich and Canny 1995] and the many variants thereof e.g., [Guendelman et al. 2003], are essentially Gauss Seidel processes and are expected to suffer from low accuracy, especially when they are limited to one or two sweeps through the constraints.

# 6 Implementation Details

The principal pivot methods of Keller and Lemke, as well as the smoothed Newton method were implemented in Octave. The other methods were implemented in C/C++, linking to BLAS, LAPACK, and GSL, the GNU Scientific Library. Wrappers where then written in C++ to link these in Octave.

The framework used for simulating rigid multibody systems was the Vortex library from CMLabs Simulations (see `http://www.cm-labs.com`).

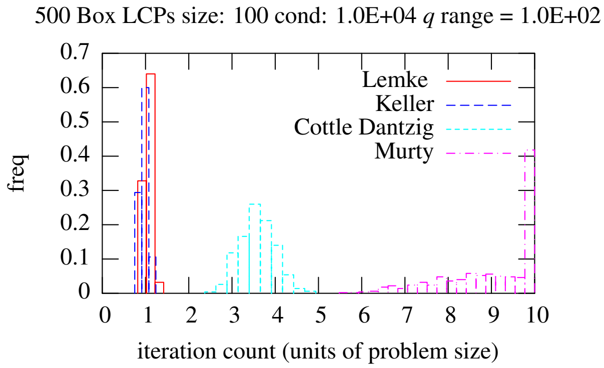


Figure 1: Histogram data for pivot methods on box LCP. Iteration count is in units of the problem size.

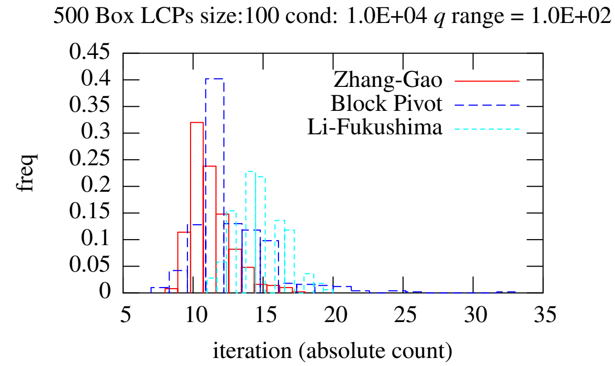


Figure 2: Histogram data for two Newton methods on box LCP of size 100. Iteration count is in absolute units.

We did not perform timing analysis for this report but concentrated on overall iteration count. This provides worse case analysis to select a viable method which will eventually be optimized.

## 7 Experimental Results

We tested box LCP solvers on a set of random problems generated using the method described in[Alefeld et al. 1999]. All matrices have full rank in these tests. For the few methods which can be started at an advanced point near the solution, we concentrated on the worse case scenario, starting with a $z$ vector at the lower bounds.

Data is presented using histogram because no LCP solver is guaranteed to process a given problem in less than $2^n$ operations. Histograms give an idea of the expected operation count for a given family of problems. Sharp peaks indicate that the given solver is nearly deterministic in terms of operation count. Broad distribution indicate that a given solver can take wildly different amounts of computation time to process different problems. These graphs represent the frequency of problems solved against iteration count. For pivot method, we used relative iteration counts, normalized to problem size. For Newton methods, the iteration counts are nearly independent of problem size and we use absolute iteration counts.

Data for random box LCPs of size 100 is shown in Fig. 1. We found sharp peaks for both Lemke's and Keller's method near $n$ pivot steps. The Cottle-Dantzig method shows a broad peak near $2n$ pivot steps and Murty's method is slightly worse. The sharp peaks remain for Lemke's and Keller's method for larger systems but the distribution for Murty's and the Cottle Dantzig methods stretch out further to higher iteration counts and become much broader as well. This suggests that Lemke's and Keller's method are near deterministic in performance. For bigger problems, Murty's and Cottle-Dantzig's methods are not usable. Murty's method can be started at an advanced point near the solution in which case it might offer better performance but the worse case scenario is not promising.

For Newton-type methods, we present two solvers: a block principal pivot method [Kostreva 1978] and a globally convergent smoothed Newton method[Zhang and Gao 2003]. For these the iteration count is expected to be independent of problem size and the distribution should be sharp. Results for box LCPs of size 100 are show in Fig. 2. The block pivot distribution is slightly skewed towards higher iteration count because it can actually cycle. Our implementation includes cycle detection and restart as described in[Júdice and Pires 1994].

Next we turn to the iterative block Gauss Seidel solver. On random problems, this behaved just as per the theory so we omit the

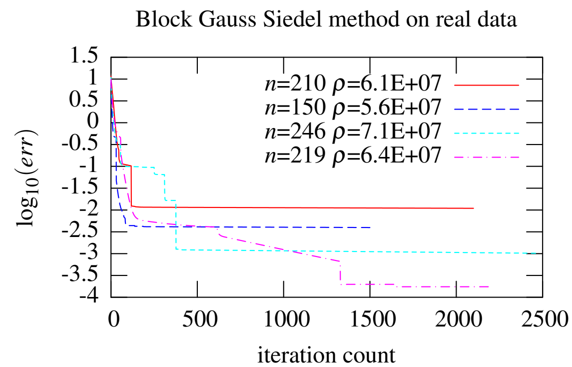


Figure 3: Residual decay as a function of iteration for a stacking problem with 40 cylinders. Each iteration is a one complete sweep over the diagonal elements.

results. Essentially, for condition numbers greater than 1E7, this is not usable. Scaling both the rows and the columns did not improve the results much. Surprisingly, when this is used to simulate stacks of rigid bodies, the anomalies are not immediately apparent despite large residual norms. This is appealing since one can quit early after a fixed number of iteration. Nevertheless, the convergence rate is too low for this to be a good alternative and the method often stagnates as is seen in Fig 3.

We used the Vortex Toolkit from CMLabs and modified the engine to use the Keller method. Using both box friction and scaled box friction model, we simulated stacking problems with 40 identical cylindrical logs falling on each other. This leads to degenerate systems. The prototype solvers were then tested on data extracted from the simulations. A still frame from the simulation is shown in Fig. 4. Results a are shown below in Fig. 5 for pivot methods and Fig. 6 for Newton methods.

Finally, we tested the convergence of the operator splitting scheme. First, for the case of a box sliding down an inclined plane, we found that a two pass scheme starting with zero friction yields the correct Coulomb relation. However, a two pass scheme starting with infinite friction doesn't behave correctly: it needs about 5 iterations to converge to the correct answer. This demonstrates that in general, a multi-pass method is necessary. Results are summarized in Fig. 7.

On a more complicated problem with a stack of 40 cylinders, we

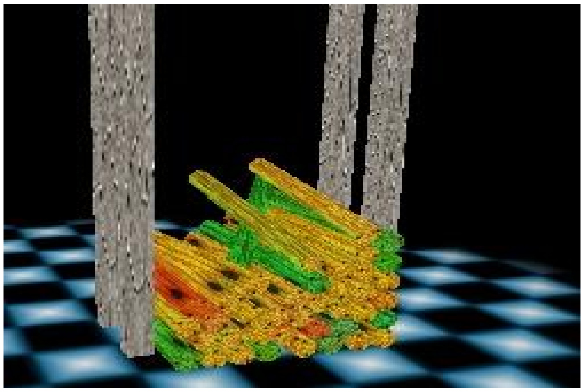

Figure 4: A stack of 40 indentical cylindrical logs falling under gravity.

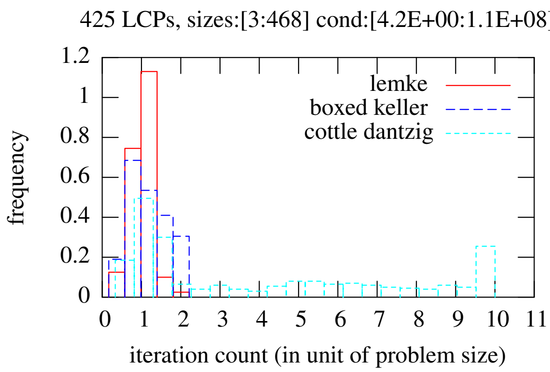


Figure 5: Histogram data for three pivot methods on box LCP problems extracted from a simulation with 40 cylinders forming a heap. The first two peaks correspond to Lemke's method which never seems to take more than $n$ iterations for these problems. Keller's method takes at most $1.2n$ iterations but Cottle-Dantzig's principal pivot method has a long tail up to $4n$ iterations.

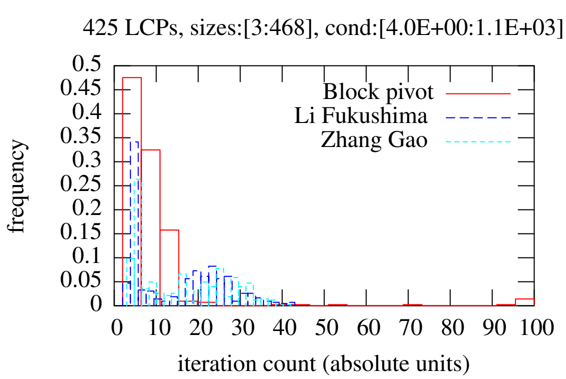


Figure 6: Histogram data for three Newton methods on box LCP problems extracted from a simulation with 40 cylinders forming a heap. A diagonal perturbation of 1E-3 was applied on the diagonal to regularize the problems.

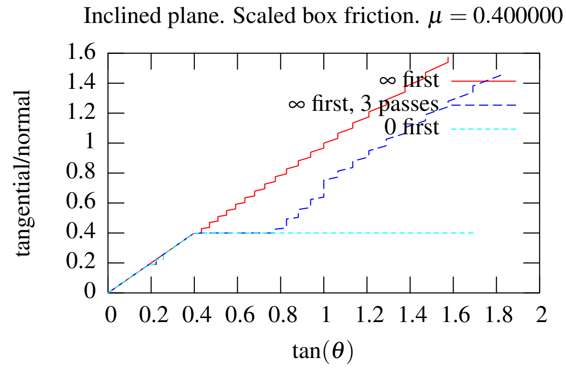


Figure 7: Inclined plane experiment with scaled box friction, using either frictionless or infinitely sticky first pass. The $y$ axis is the ratio of tangential to normal forces and the $x$ axis is the tangent of the angle of the inclined plane.

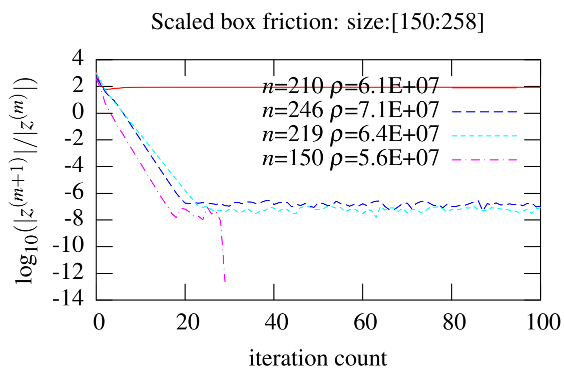


Figure 8: Convergence data for the operator splitting scheme using a smoothed Newton method for each problem. The error norm is the difference between the iterates.

found the splitting scheme could converge to a fixed point within less than 10 iterations as shown on Fig. 8, but not in all cases. The convergence rate of the splitting scheme is linear which is not the best news. Also, there appears to be stagnation after an error of 10E-8 is reached. This should be investigated further.

## 8 Conclusion and Future Work

We have shown that an operator splitting method based on box MLCP solvers is a viable alternative for solving dry frictional contact problems in rigid multibody systems. For this to work, we need a fast and robust solver for box MLCPs. Some classical pivot methods are robust and relatively efficient but not appropriate to solve really large systems as they cannot use advanced starting points or iterative techniques. Newton methods appear to be robust but more work is required to make them efficient. In particular, more research is needed to speed up the computation of the search direction. The performance results also indicate that popular solvers such as the Cottle-Dantzig principal pivot method and the block Gauss Seidel iterative method adapted for solving MLCP are far from optimal choices when it comes to efficiency or accuracy.

The relevance to interactive graphics is that a robust method which is guaranteed to work with a more or less guaranteed bound on the amount of work to update a system with a given number of bodies and constraints is essential for the context of real-time interactive applications. We have identified solvers for MLCP with box constraints which meet these requirements though at this time, the expected complexity is still too high at $O(n^3)$, where $n$ is the total number of constraints.

More work is also needed to improve the convergence rate of the splitting method. It might be possible to include extra equations in a smoothed Newton scheme to make the convergence rate quadratic instead of linear as is reported here. Box friction models are notoriously anisotropic, a problem which was not presented above but which we intend to address as well.

## 9 Acknowledgments

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