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Probabilistic Variation Mode and Effect Analysis: A Case Study of an Air Engine Component

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Introduction

An important goal of engineering design is to get a reliable system, structure or component. One such well-established method is FMEA (Failure Mode and Effect Analysis), where the aim is to identify possible failure modes and evaluate their effect. A general design philosophy, within robust design, is to make designs that avoid failure modes as much as possible, see e.g. (Davis, 2006). Further, it is important that the design is robust against different sources of unavoidable variation. A general methodology called VMEA (Variation Mode and Effect Analysis) has been developed in order to deal with this problem, see (Johansson, et al., 2006) and (Chakhunashvili, et al., 2006). The VMEA is split into three different levels; 1) basic VMEA, in the early design stage, when we only have vague knowledge about the variation, and the goal is to compare different design concepts, 2) advanced VMEA, further in the design process when we can better judge the sources of variation, and 3) probabilistic VMEA, in the later design stages where we have more detailed information about the structure and the sources of variation, and the goal is to be able to asses the reliability.

This paper treats the third level, the probabilistic VMEA, and we suggest a simple model, also used in (Svensson, 1997), for assessing the total uncertainty in a fatigue life prediction, where we consider different sources of variation, as well as statistical uncertainties and model uncertainties. The model may be written as

$$Y = \hat{Y} + X_1 + X_2 + \ldots + X_p + Z_1 + Z_2 + \ldots + Z_q \qquad (1)$$

where the X_k's and the Z_k's are random variables representing different sources of scatter or uncertainty, respectively. In our case it is appropriate to study the logarithmic life, thus $Y = \ln N$, where N is the life. The prediction of the logarithmic life $\hat{Y} = \ln \hat{N}$ may be a complicated function, e.g. defined through finite element software; however the analysis of the prediction uncertainty is based on a linearization of the function, making use of only the sensitivity coefficients. Further, for reliability assessments the log-life, $Y = \ln N$, is approximated by a normal distribution. The methodology will be discussed using a case study of a low pressure shaft in a jet engine, see Figure 1.

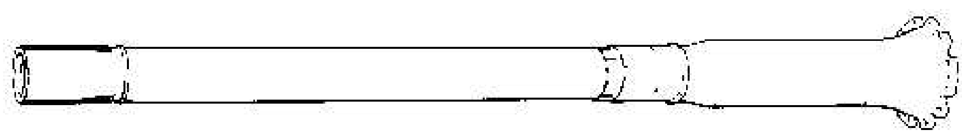

Figure 1. Low pressure shaft in a jet engine.

Scatter & uncertainty

There are various ways in which the types of variation might be classified, see e.g. (Melchers, 1999), (Ditlevsen & Madsen, 2005) and (Lodeby, 2000). The first way is to distinguish between *aleatory* uncertainties and *epistemic* uncertainties. The first one refers to the underlying, intrinsic uncertainties, e.g. the scatter in fatigue life and the variation within a class of customers. The latter one refers to the uncertainties which can be reduced by means of additional data or information, better modelling and better parameter estimation methods.

Here we will use the terminology *scatter* as being the aleatory uncertainties, and just *uncertainty* as being the epistemic uncertainties. In our approach, we will focus on the three kinds of uncertainties mentioned by (Ditlevsen & Madsen, 2005), and denote them by

- ***Scatter*** or physical uncertainty is that identified with the inherent random nature of the phenomenon, e.g. the variation in strength between different components.
- ***Statistical uncertainty*** is that associated with the uncertainty due to statistical estimators of physical model parameters based on available data, e.g. estimation of parameters in the Coffin-Manson model for life based on fatigue tests.
- ***Model uncertainty*** is that associated with the use of one (or more) simplified relationship to represent the 'real' relationship or phenomenon of interest, e.g. a finite element model used for calculating stresses, is only a model for the 'real' stress state.

Another important kind of uncertainties is what can be called call uncertainty due to human factors. These are not treated here, but must be controlled by other means, which is discussed in e.g. (Melchers, 1999).

A simple approach to probabilistic VMEA

Here we will present a simple model for the prediction uncertainty based on a summation of contributions from different sources on scatter and uncertainty. We will discuss it in terms of fatigue life prediction, but it can easily be adapted to other situations, e.g. prediction of maximum stress or maximum defect sizes.

Model for uncertainty in life predictions

We will study the prediction error of the logarithmic life prediction

$$\hat{Y} = \ln \hat{N} = f(\Psi, \hat{\boldsymbol{\theta}}, \hat{\mathbf{X}}) \tag{2}$$

where $f(\Psi, \hat{\boldsymbol{\theta}}, \hat{\mathbf{X}})$ is our model for the life which involves the damage driving parameter, Ψ (e.g. stress, strain or force), the estimated parameter vector, $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \ldots, \hat{\theta}_r)$, and the modelled scatter, $\hat{\mathbf{X}}$. The prediction error can be written as

$$\hat{e} \quad = \quad Y - \hat{Y} = \ln N - \ln \hat{N} = g(\Psi, \tilde{\mathbf{X}}) - f(\Psi, \hat{\boldsymbol{\theta}}, \hat{\mathbf{X}}) \tag{3}$$

where $g(\Psi, \tilde{\mathbf{X}})$ is the actual relation for the log-life depending on the damage driving parameter, Ψ, and the involved scatter, $\tilde{\mathbf{X}}$. The next step is to approximate the prediction error by a sum

$$\hat{e} \quad = \quad X_1 + X_2 + \ldots + X_p + Z_1 + Z_2 + \ldots + Z_q \tag{4}$$

where the quantities $\mathbf{X} = (X_1, \ldots, X_p)$ and $\mathbf{Z} = (Z_1, \ldots, Z_q)$ represent different types of scatters and uncertainties, respectively. The random quantities X_i and Z_j are assumed to have zero mean and variances τ_i^2 and δ_j^2, respectively. In the analysis we only use the variances and covariances of the X_i's and Z_j's, but not their exact distributions, which are often not known to the designer. In some situations it is more natural to estimate the scatter or uncertainty in some quantity that is related to log-life, and then use a sensitivity coefficient to get its effect on the log-life. One such example is the scatter in life due to geometry variations due to tolerances, where it is easier to estimate the scatter in stress, say τ_i', which is then transferred via a sensitivity coefficient to the scatter in life, $\tau_i = |c_i| \cdot \tau_i'$. This is motivated by Gauss approximation formula for the transfer function $\ln N = h(S)$ from stress to log-life, which gives

$$\ln N = h(S) \approx h(s_0) + c_i (S - s_0) \quad \text{with} \quad c_i = \left. \frac{dh}{dS} \right|_{S=s_0} \tag{5}$$

where s_0 is the stress corresponding to the nominal values.

When constructing reliability measures, we will make use of a normal distribution approximation on the log-life. The logarithmic transformation used here has an important implication. Namely, the variation measures τ_i^2 and δ_j^2 can be interpreted as variation coefficients for the life

$$\mathrm{Var}[\ln N] \approx \frac{\mathrm{Var}[N]}{\mathrm{E}[N]^2}. \tag{6}$$

This interpretation is practical when one is forced to use engineering judgements for estimates of uncertainties, since they can easily be related to percentage uncertainty. In the industrial example below we will consider life prediction for low cycle fatigue, where the damage driving parameter is the strain range Δε. The life model in this case is the Basquin-Coffin-Manson equation

$$\frac{\Delta\varepsilon}{2} = \frac{\sigma_f'}{E} \cdot (2N)^b + \varepsilon_f' \cdot (2N)^c \tag{7}$$

which in cases of large strain ranges can be simplified to the Basquin-Coffin-Manson equation

$$\frac{\Delta\varepsilon}{2} \approx \varepsilon_f^{'} \cdot (2N)^c \qquad (8)$$

which can be rewritten to the

$$N = e^a \cdot \Delta\varepsilon^{1/c} \quad \Leftrightarrow \quad \ln N = a + \frac{1}{c}\ln\Delta\varepsilon \quad \text{with} \quad a = -\ln 2 - \frac{1}{c}\ln(2\varepsilon_f^{'}) \qquad (9)$$

giving the life prediction model $\ln\hat{N} = f(\Psi, \hat{\boldsymbol{\theta}}, \hat{\mathbf{X}})$ as a linear regression model, which will be used in our case study.

Estimation of prediction uncertainty

There are different kinds of uncertainties that need to be estimated. For scatter, the straight forward method is to make an experiment and calculate the sample standard deviation. In more complicated situations ANOVA (Analysis of Variance) is a useful tool. However, it is not always possible or economically motivated to perform experiments. Instead informed guesses, previous designs, or engineering experience have to be used. Concerning statistical uncertainty, there are standard statistical methods like maximum likelihood theory for finding expressions of the variances of the estimates. In more complicated situations the bootstrap method can be applied. However, in order to use these statistical methods it is required that the original data is available, which is not always the case. A typical example is that only the estimated life curve is available, and maybe also information about the number of tests performed. In these situations it is also necessary to have a method in order to get an idea of the statistical uncertainty. For a specified model the model uncertainty is in fact a systematic error. However, if we consider the prediction situation such that we randomly choose a model from a population of models, then the systematic model error appears as a random error in the prediction. We will discuss different methods for estimating the model uncertainty, e.g. considering a random choice of models from the population of models, or considering extreme cases of models. In the following, we will give some more details and examples on the estimation, and motivate the estimated values in Table I.

Life prediction	**logarithmic life, ln(N)**		
Type of scatter & uncertainty	scatter	uncertainty	total
Strength scatter			**0.38**
- Material, within shaft	0.15		
- Material, between shafts	0.29		
- Geometry	0.20		
Statistical uncertainty			**0.07**
- LCF-curve		0.07	
Model uncertainty			**0.84**
- LCF-curve		0.05	
- Mean stress model		0.30	
- Multi- to uni-axial		0.20	
- Plasticity		0.72	
- Stress analysis		0.24	
- Temperature		0	
Load scatter & uncertainty			**0.50**
- Service load, scatter	0.40		
- Service load, uncertainty		0.30	
Total	**0.55**	**0.90**	**1.05**

Table I. Table summarizing the sources of scatter and uncertainty and their contributions, in terms of standard deviation of the logarithmic life, to the total prediction uncertainty.

Estimation of scatter

In our case study, the evaluation of the scatter is based on fatigue tests of three different shafts resulting in six observed lives each. From an ANOVA it was found that there is both a within shaft scatter and a between shafts scatter, estimated at $\tau_{within} = 0.15$, and $\tau_{between} = 0.29$, respectively, see Figure 2. The within shaft scatter originates from material scatter, whereas the between shafts scatter is production scatter due to different batches, processing, or supplier effects.

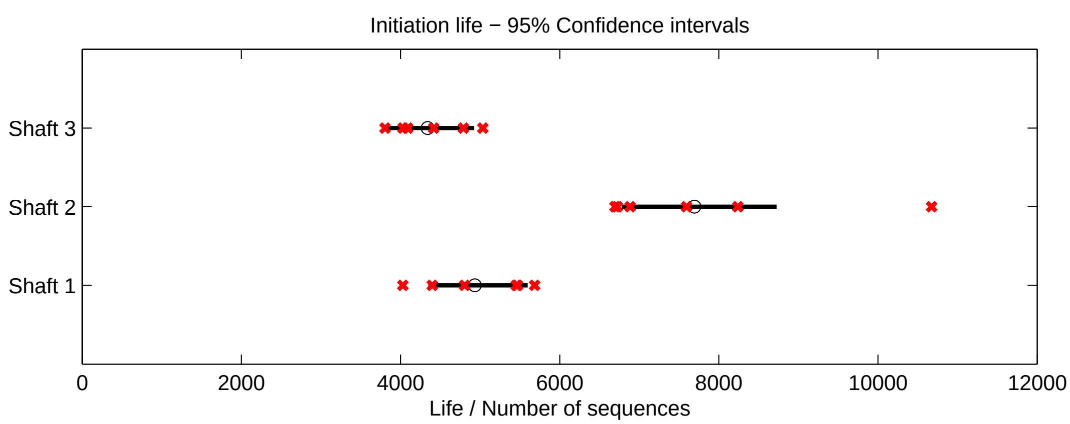

Figure 2. Confidence intervals for median life for the three different shafts tested.

In many situations it is necessary to rely on engineering judgement, and for example ask a man in the work shop “what is the worst case?”. The answer should often not be interpreted as representing zero probability of observing something more extreme, but rather that there is a very small risk of that. Therefore, the ‘worst case’ statement can be mathematically interpreted as a certain quantile, e.g. the 1/1000 risk of observing a more extreme case. By further assumptions, say a normal distribution, it is possible to estimate the scatter, τ, according to

$$z_{0.001} = \mu + 3\tau \quad \Rightarrow \quad \tau = (z_{0.001} - \mu)/3 \tag{10}$$

where μ is the nominal value, $z_{0.001}$ is the ‘worst case’ representing the 0.1% quantile, and the value 3 comes from the 0.1% quantile of the standard normal distribution.

The strength of a structure and a component in the structure depends not only on the material properties, but is also highly dependent on geometry and assembling quality. For the example in Table I, the geometry was varied according to the tolerances and the worst case resulted in a 10% change of the calculated stress. The worst case was here interpreted as the 0.1% quantile. Further, the sensitivity coefficient from stress to log-life was estimated at -6, giving the estimated scatter, $\tau = 6 \cdot 0.10/3 = 0.20$.

Statistical uncertainty

There are standard statistical methods like maximum likelihood theory for finding expressions of the variances of estimates, see e.g. (Casella & Berger, 2001) or (Pawitan, 2001). For more complicated estimation procedures the bootstrap method, which is a general method based on simulations, can be applied, see e.g. (Efron & Tibshirani, 1993), (Davison & Hinkley, 1997), and (Hjort, 1994).

In case of the Basquin-Coffin-Manson model (9), the parameters may be estimated using linear regression. In this situation the prediction uncertainty is

$$s\sqrt{1 + \frac{1}{n} + \frac{(x - \bar{x})^2}{\sum_i (x_i - \bar{x})^2}} \approx s\sqrt{1 + \frac{1}{n} + \frac{1}{n}} = s\sqrt{1 + \frac{2}{n}} \tag{11}$$

where x_i is the logarithmic strain during reference test i, x is the predicted logarithmic service strain, and the last expression is a rough approximation of the root expression. This simple approximation is correct if the squared distance from the actual value x to the reference test mean level $\bar{x}$ is the same as the mean square distance in the reference tests. In case of interpolation use of the model this approximation is usually good enough. The simplification (11) can be extended to models with more variables using the expression

$$s\sqrt{1 + \frac{r}{n}}, \tag{12}$$

where *r* is the number of parameters in the model. This kind of approximation is especially useful in cases where the original data is not available, but only the estimated life curve together with the observed experimental scatter and the number of tests performed. Thus, in these situations, a rough guess of the statistical uncertainty can be obtained as $\delta = s\sqrt{r/n}$. This was used for the case study, where a four-parameter Coffin-Manson curve from literature was used, which was based on 20 tests. Thus, the statistical uncertainty is estimated as $\delta = 0.15\sqrt{4/20} = 0.07$.

Model uncertainty

Assume that we have made life predictions based on different models, and want to address the uncertainty arising from the choice of model. We will consider two situations:

1. Assume that there is one model representing the least favourable case, and another representing the most favourable case. This means that these two models represent extreme cases of models, and all other models predict lives somewhere in between.
2. Assume that the models have been arbitrarily chosen; in other words they are randomly chosen from a population of models.

In the first situation, without any other information, it is natural to assume a uniform distribution with the end-points according to the extreme cases. This gives an estimate, based on the uniform distribution, of the standard deviation

$$s_U = \frac{\ln L_{\max} - \ln L_{\min}}{2\sqrt{3}} \qquad (13)$$

with $L_{\min}$ being the shortest predicted life, and $L_{\max}$ the longest.

In the second situation, we may use the sample standard deviation as estimator

$$s = \sqrt{\frac{1}{n-1}\sum_{k=1}^{n}\left(\ln L_k - \overline{\ln L}\right)^2}, \qquad \overline{\ln L} = \frac{1}{n}\sum_{k=1}^{n}\ln L_k \qquad (14)$$

where we have used n different life models predicting the lives $L_1, L_2, \ldots, L_n$.

For very long load sequences (the normal situation for turbojet engine components) it is normal procedure to perform linear elastic finite element calculations. Plasticity will therefore have to be handled by plasticity models. For the plasticity model, the Linear rule and the Neuber rule for plastic correction can be seen as two extreme cases of models. In a similar situation the predictions using the two models differed by more than a factor two, and in that case the model uncertainty was estimated to be

$$\delta_{plasticity} = \frac{1}{2\sqrt{3}}(\ln 21 - \ln 1.77) = 0.72\,. \qquad (15)$$

The model errors due to mean stress correction and the conversion from multi-axial to uniaxial stress state, where analyzed in the same manner. Further, the stresses were calculated using finite element programs and the uncertainty in the calculated stresses were judged to be about 4%. Thus, using the sensitivity coefficient of 6, the uncertainty in log-life is $\delta_{stress} = 6 \cdot 0.04 = 0.24$. Often in jet engine applications the there are effects of high temperature. In this case this model error was judges to be negligible, hence $\delta_{temperature} = 0$.

Scatter and uncertainty in loads

The load that a component will experience during its time in service is usually very difficult to estimate. Experience from measurements in service for certain predefined manoeuvres gives a rough estimate of the expected load scenario, but variations and extreme events should also be considered.

Often this problem is hidden from the designer, since demands are already given by other departments in the company. However, it is important to estimate the load scatter and uncertainty, since this part of the load/strength problem may override other parts and make the variability in strength negligible.

In our example, the estimates are based on previous engineering experience. There is a scatter in the load due to the individual usage, which was judged to be $\tau_{load} = 0.40$. There is also an uncertainty whether the flight missions used for design really reflect the typical usage in field, where the uncertainty was judged to be $\delta_{load} = 0.30$.

Total prediction uncertainty

The total prediction uncertainty is the sum of all the contributions; see Eq. (4),

$$\delta_{pred} = \sqrt{\tau_1^2 + \tau_2^2 + \ldots + \tau_p^2 + \delta_1^2 + \delta_2^2 + \ldots + \delta_q^2} \tag{16}$$

which is the number in the right most bottom corner of Table I. In this example it was reasonable to assume independence between the different sources of scatter and uncertainty, hence no covariance terms appear in Eq. (16). However, in case of correlation between the scatters and/or uncertainties, it is simply to add these covariance terms under the root sign in formula (16).
In the following section we will describe how the result can be used for life assessment. Another important use is to get information on where is it most efficient to try to reduce variation. In our case, the model uncertainty due to plasticity is by far the largest uncertainty, and thus it is motivated to further study the plasticity, in order to reduce its model uncertainty.

Reliability assessment

We will now demonstrate how the estimated total prediction uncertainty in log-life can be used and presented as a prediction interval for the life or as a safety factor for the life. The analysis is based on the construction of a prediction interval using a normal approximation

$$\ln N = \ln N_{pred} \pm z_p \cdot \delta_{pred} \quad \Rightarrow \quad N = N_{pred} \cdot \exp(\pm z_p \cdot \delta_{pred}) \tag{17}$$

where N_{pred} is the life prediction according to the calculation, δ_{pred} is our previously estimated total prediction uncertainty in log-life. The factor z_p is a quantile of the standard normal distribution; $z_{0.025} = 2.0$ for a 95% interval, and $z_{0.001} = 3.0$ for a 99.8% interval.
In Figure 3 the 95% prediction interval for the calculated life prediction according to Eq. (17), are compared to the ones obtained from fatigue life tests.

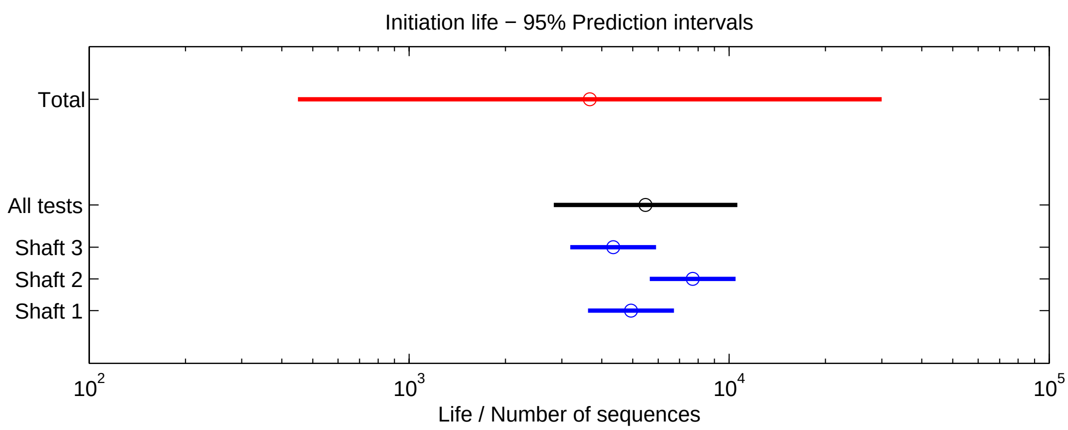

Figure 3. Prediction intervals for life, log-scale.

The experiments revealed a scatter between the shafts. The experimental prediction intervals are therefore shown for each shaft individually as well as for the total scatter (within and between), compare Figure 2. Note that the experimental life interval can be seen as a lower limit of the length of the prediction intervals, since it involves only the scatter in life that is unavoidable. The experiments could be used to update the calculation of the life prediction, i.e. to correct for the systematic error and reduce some of its uncertainties, especially the model uncertainties. This has not been pursued for this case study.
For design the lower endpoint of the prediction interval should be considered in order to get a reliable component. Often a reliability corresponding to a risk of 1/1000 is used. In Table II the limits of the 95% and 99.8% prediction intervals are presented.

Initiation life	Quantiles				
	0.1%	2.5%	50%	97.5%	99.9%
Life prediction	160	450	3 700	30 000	86 000

Table II: Prediction quantiles for log-life.

It is possible to define a safety factor in life, based on the prediction interval, as the ratio between the median life and a low quantile of life

$$K_p = \frac{N_{0.5}}{N_p} \tag{18}$$

where p is the probability of failure. In our case it become

$$K_p = \exp(\ln N_{0.5} - \ln N_p) = \exp(z_p \cdot \delta_{pred})\,. \tag{19}$$

For the initiation life the safety factors in life becomes $K_{0.025} = 8.2$ and $K_{0.001} = 23$.

Conclusions and discussions

In the early design stages the basic and enhanced VMEA should be used, and based on the result of these, the most critical components are identified. For some components it may then be motivated to make a more sophisticated probabilistic VMEA, where the involved scatter and uncertainties need to be quantified. At first this can be made in quite a rough way, making use of the available information in the design process, but also making use of previous experience from similar structures, and other kinds of engineering experience. The goal is to get a rough estimate of the prediction uncertainty, and to locate the largest sources of scatter and uncertainty, in order to see where further efforts would be most efficient. The proposed method represents the concept of First-Order Second-Moment (FOSM) reliability theory, see e.g. (Melcher, 1999) or (Ditlevsen & Madsen, 1996). The First-Order refers to the linearization of the objective function, and the Second-Moment refers to the fact that only the means and variances are used.
In the present case study, the largest contribution to the prediction uncertainty originates from the model uncertainty due to the modelling of the plasticity. Therefore, in this specific case, it is motivated to further study the plasticity phenomenon. One may discuss whether the judgement of the uncertainty is realistic, and if the size of the contribution may be overestimated. However, it turns out that the model uncertainty is realistic, and it is motivated to model the plasticity phenomenon in more detail. With the computer performance of today, still increasing, it becomes more and more realistic to perform non-linear analyses. A load sequence containing some hundred load steps is today realistic to evaluate with a non-

linear material model in the finite element calculation. The model uncertainty due to plasticity can then be expected to drop from 0.72 to 0.20, which would result in a reduction of the prediction uncertainty from 1.05 to 0.79.

Another example on the influence from tolerances was found on a similar component. The critical point of this component was a u-shaped groove. The influence from tolerances on the fatigue life in the u-groove was entirely depending on the tolerances of the radius in the u-groove. A change of this radius, from minimum to maximum radius within the tolerance zone, resulted in a fourfold variation of fatigue life.

The split of the uncertainty sources in scatter and uncertainty gives possibilities to update the reliability calculation in a rational manner when new data are available. In case of experiments made under similar conditions as in service, the model errors may be estimated and corrected for. In case of measurements of service loads the corresponding uncertainty entry in Table I can be updated.

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