A control chart of the Weibull percentiles via Bayesian - bootstrap approach

Pasquale Erto
ertopa@unina.it
University of Naples "Federico II"
Naples, Italy

Giuliana Pallotta
g.pallotta@unina.it
University of Naples "Federico II"
Naples, Italy

ABSTRACT

Purpose: This work proposes an innovative control chart of the Weibull percentiles using Bayesian estimators supported by bootstrap methods.

Approach: The chart offers two main advantages.

On one side, the estimation procedure is able to effectively integrate both the experimental and the technological information exploiting some specific Bayesian estimators.

On the other side, the bootstrap techniques allow to capitalize the experimental information provided by few samples.

Findings: The performance of the control chart has been investigated by means of a large Monte Carlo study.

Value of the paper: The paper presents a control chart for Weibull percentiles, where few alternative charts can be found.

Keywords: Statistical Process Control, non-Normal control charts, Bayesian inference, Weibull distribution, Bootstrap methods

Category: Research Paper

INTRODUCTION

The common Shewhart-type control charts are widely employed to monitor quality of products and services in order to detect shifts (step changes or gradual drifts) in the mean and/or variance of a quality characteristic of interest. However, these charts work satisfactorily if the underlying distribution of the observed data is Normal or near-Normal and the sample size is large enough. This assumption allows one to exploit the "normalizing effect" and to theoretically derive the sampling distribution of the parameter estimator.

Anyway, this assumption is not valid in several technological contexts where the quality is measured in terms of reliability. Nearly always, the distributions of the reliability parameters are skewed and the "normalizing effect" of Shewhart control charts is not effective or impossible due to the extremely small size of the available samples. Unfortunately, few papers dealing with non-Normal populations and reliability control can be found in literature (Padgett and Spurrier, 1990), (Kanji and Arif, 2001), (Xie *et al.*, 2002), (Shore, 2004), (Zhang and Chen, 2004), (Nichols and Padgett, 2005).

Moreover, in these alternative charts, reliability control is performed via a pure classical statistical approach, ignoring all the available technological knowledge. Anyway, in facing very small samples, these classical estimation procedures can be misleading as shown in (Canavos and Tsokos, 1972, 1973), (Smith and Naylor, 1987) and (Erto, 2005).

To overcome all these difficulties we can use reliability estimators based on the application of the Bayes theorem, that has got the peculiarity of allowing one to integrate both the experimental and the technological information.

Moreover, within a Bayesian framework bootstrap techniques can be used to support the estimation procedure, in order to capitalize the experimental information provided by few samples.

The control chart proposed in this work exploits some specific Bayesian reliability estimators known as "Practical Bayes Estimators" (PBE) in order to monitor a specified percentile of the underlying Weibull distribution of the characteristic of interest. These estimators were first introduced in (Erto, 1982) and then applied in several technological contexts during past years (Erto and Rapone, 1984), (Erto and Lanzotti, 1990), (Erto and Giorgio, 1996), (Erto, 2005), (Erto and Pallotta, 2006), (Erto and Pallotta, 2007) and (Erto *et al.*, 2008).

In the first sections the main steps are presented which led to estimate the sampling distribution of the Bayesian estimator for the Weibull percentile by means of the "parametric bootstrap sampling". Then this empirical sampling distribution is exploited to draw a Shewhart-type control chart for Weibull percentiles. The performance of the chart is investigated by means of a large Monte Carlo study and a comparative analysis of its responsiveness is reported.

Acronyms & Notation

$\mathrm{Sf}\{\cdot\}$	survival function
$\mathrm{pdf}\{\cdot\}$	probability density function
R	reliability level
K	constant equal to $\ln(1/R)$
x_R	$1-R$-th percentile of the Weibull distribution such as that $\mathrm{Sf}\{x_R\}=R$
$E\{x_R\}$	anticipated (mean) value given by prior knowledge for x_R
n	sample size
$\underline{x}$	n-dimensional sample array
k	number of available samples
M	number of bootstrap training samples
B	number of pseudo-random samples from estimated Weibull distribution
α	tail areas of a pdf corresponding to a given false alarm risk
δ, β	scale and shape parameters of the Weibull distribution
a, b	scale and shape parameters of the Inverse Weibull prior distribution of x_R
β_1, β_2	prior numerical interval for β
$\hat{}$	implies an estimate.
PBE	Practical Bayes Estimators (or Estimates)
L	Likelihood Function
MLE	Maximum Likelihood Estimators (or Estimates)

1 THE RELEVANCE OF THE WEIBULL PERCENTILES IN RELIABILITY CONTROL

In reliability problems, the control of the Weibull percentile x_R is strategic. The choice of x_R is justified by the wide use of this parameter in:

- defining warranty conditions;
- developing contractual specifications;
- characterizing norm requirements;
- expressing key-indicators in engineering catalogues.

In practical situations the control of a specific Weibull percentile becomes crucial when the quality characteristic of interest is the breaking strength of brittle materials (such as carbon, boron), the compressive strength of specimens made with quasi-brittle materials (such as concrete, rock, ice, ceramic and composite materials concrete) and the tensile adhesive strength (Park, 1996). In these contexts, a minimum strength value is required for engineering design and monitoring the mean and variance of the strength distribution (by means of classical control charts) could be seriously less effective than monitoring lower percentiles since a small variation in mean and/or variance can produce a significant shift in the small percentile of interest as shown in (Padgett and Spurrier, 1990) and (Nichols and Padgett, 2005).
Moreover, we must outline that, if the reliability level of the tested items is very high, we are able to collect very few data which prevent us to use classical control charts. In these cases, the proposed Bayesian approach may result an appropriate one.

The Weibull survival function is:

$$\mathrm{Sf}\left\{x;\delta,\beta\right\}=\exp\left[-\left(x/\delta\right)^{\beta}\right];\qquad x\geq 0;\qquad \delta,\beta>0 \tag{1}$$

that can be immediately reparameterized in terms of the percentile x_R and shape parameter, β, in which the Engineers' knowledge can be more easily converted:

$$\mathrm{Sf}\left\{x;x_R,\beta\right\}=\exp\left[-K\left(x/x_R\right)^{\beta}\right];\qquad x\geq 0;\qquad x_R,\beta>0 \tag{2}$$

$$K=\ln\left(1/R\right)$$

x_R and β both being unknown.

In the above engineering context, very good knowledge exists about the mechanism of failure under consideration, which can be converted into quantitative form about β. In particular, the engineers working in these well known contexts usually know more than the simple order of magnitude of the reliability performance which the produced item has, e.g., he has a quite precise knowledge about an x_R. Then, with both these pieces of information, he can formulate a numerical interval $\left(\beta_1,\beta_2\right)$ for β and an anticipated value for x_R. The PBE allow combining this prior knowledge about β and x_R with a few experimental data to give very good Weibull parameter estimates.

2 THE BAYES APPROACH IN RELIABILITY PROBLEMS

2.1 Technological knowledge and prior distributions

In order to evaluate the PBE of the Weibull parameters, the following main elementary steps must be performed. For a comprehensive discussion of the assumptions on which the resulting estimates are based see (Erto, 1982) and (Erto, 2005).

The prior knowledge for β is converted into the values β_1 and β_2, using the well known relationship between the mechanism of failure and the value of β (e.g., early failures imply $\beta < 1$; chance failures imply $\beta = 1$; wear out failures imply $\beta > 1$).

These two parameters can be effectively and easily anticipated since the Engineers' information, about the mechanism of failure, can be always expressed in terms of an interval (β_1, β_2). This interval must be chosen wide enough in order to plausibly contain the unknown (true) value of the Weibull shape parameter.

The only restriction, which the anticipated values β_1 and β_2 must be subjected to, is the following one:

$$\beta_1 + \beta_2 > 2 \tag{3}$$

since their sum will be used to set up the argument of the Gamma function in (6).

For a selected percentile x_R (corresponding to the R value of interest) the prior probability density function is assumed to be the Inverse Weibull (Johnson *et al.*, 1994):

$$\text{pdf}\{x_R\} = a\,b\,(a\,x_R)^{-(b+1)} \exp\left[-(a\,x_R)^{-b}\right]; \qquad x_R \ge 0; \qquad a, b > 0 \tag{4}$$

where a and b are scale and shape parameters respectively. The prior information for x_R is converted into the mean value of the probability density function (4):

$$E\{x_R\} = \frac{\Gamma(1 - 1/b)}{a} \tag{5}$$

Then, an effective value for the prior parameter a is automatically obtained by means of the (5):

$$a = \frac{\Gamma(1 - 1/\beta_m)}{E\{x_R\}}; \qquad \beta_m = (\beta_1 + \beta_2)/2 \tag{6}$$

since it is reasonable to assume $b = \beta_m$, as discussed in (Erto, 1982).

2.2 Reliability tests and Practical Bayes Estimators

Usually, in reliability tests, a sample array $\underline{x}$, of n experimental data, is available. If the selected reliability parameters of the items are characterized by the model (2), the following joint posterior probability density is obtained:

$$\text{pdf}\{x_R, \beta \,|\, \underline{x}\} = \frac{\beta^{n+1}\, a^{-\beta}\, x_R^{-\beta(n+1)-1} \prod_{i=1}^{n} x_i^{\beta-1} \exp\left[-x_R^{-\beta}\left(a^{-\beta} + K \sum_{i=1}^{n} x_i^{\beta}\right)\right]}{n! \int_{\beta_1}^{\beta_2} \beta^n a^{-\beta} \prod_{i=1}^{n} x_i^{\beta-1} \left(a^{-\beta} + K \sum_{i=1}^{n} x_i^{\beta}\right)^{-(n+1)} d\beta} \tag{7}$$

We can say that this density function describes the residual uncertainty which exists about the two parameters. So we could estimate the parameters x_R and β adopting their modal or median or mean values. The PBE choose the last ones, that is, the expectations of x_R and β:

$$\mathrm{E}\{x_R \mid \underline{x}\} = \frac{I_3}{I_1}; \qquad \mathrm{E}\{\beta \mid \underline{x}\} = \frac{I_2}{I_1} \tag{8}$$

where:

$$I_j = \int_{\beta_1}^{\beta_2} \beta^{m_j} a^{-\beta} \prod_{i=1}^{n} x_i^{\beta-1} \left(a^{-\beta} + K \sum_{i=1}^{n} x_i^{\beta} \right)^{-(n+1)+k_j} \Gamma\left(n+1-k_j\right) d\beta \tag{9}$$

$$j = 1, 2, 3$$

with the following values for the parameters m_j and k_j:

$$m_1 \equiv m_3 = n; \quad m_2 = n+1; \quad k_1 \equiv k_2 = 0; \quad k_3 = 1/\beta. \tag{10}$$

3 BOOTSTRAP TECHNIQUES APPLIED TO CONTROL CHARTING

During past years, bootstrap methods have gained an increasing acceptance in Statistical Process Control charting, being mainly used to find more appropriate control limits when the distribution of the observed process is unknown or non-Normal (Nichols and Padgett, 2005). In control charting the main effect of non-Normality is the difficulty to theoretically derive the sampling distribution of the selected parameter estimator. The problem is quite crucial since an appropriate knowledge about the sampling distribution influences the correct setting of control limits, according to the desired false alarm risk α (Wood *et al.*, 1999). Bajgier (1992) proposed a non-parametric bootstrap control chart to monitor process mean in case of non-Normal populations. He does not assume a distribution model but only a stable and in control process when the control limits are computed.

In (Seppala *et al.*, 1995) a bootstrap technique based on subgroups is proposed, removing the assumptions made in (Bajgier, 1992) and computing control limits using bootstrapped residuals. The available alternatives show the convenience in exploiting bootstrap methods to set up more realistic Shewhart-type control charts, able to correctly detect process shifts when the underlying distribution of the quality characteristic of interest is non-Normal.

Within this framework, a stream of research has been developed to effectively combine Bayesian estimation procedures and bootstrap techniques starting from (Laird and Luis, 1987) and (Carlin and Gelfand, 1991). The bootstrap is applied to introduce the uncertainty related to prior distributions which are estimated from data.

This approach is known as “parametric bootstrap sampling” and the control chart proposed in this paper can fit in it.

3.1 A PBE-bootstrap control chart of the Weibull percentiles

In order to construct the PBE-bootstrap control chart, obtained by integrating the PBE with the “parametric bootstrap sampling” approach, the following operative steps are presented:

Phase 1 - The training resampling procedure

1. From an in-control process, we collect k samples with sample size n. We assume that samples come from a Weibull distribution (2) with unknown parameters x_R and β. Some specific Weibull probability plots and goodness-of-fit tests (Shapiro, 1990) can be employed to check this assumption. We can decide which samples are to be

used to estimate the control limits and, as suggested in (Wood *et al.*, 1999), we could remove those judged as not typical of the process (on the basis of an earlier trial run of resampling procedure). However, in this work, we use all the k samples.

2. We pool all the observations into a single "combined sample" and, thanks to the bootstrap approach, we use it as a surrogate for the Weibull population. We assume that the "combined sample" provides an adequate picture of observations from the process and that the process is stable.
3. We resample from the "combined sample" M times obtaining M "resamples" of size n. It is generally sufficient $M = 1000$.
4. For each resample, we obtain an x_R estimate $\hat{x}_{R,j}$ and a β estimate $\hat{\beta}_j$ $(j = 1,...,M)$, using some classical estimators for Weibull parameters (such as MLE). Alternatively, the first and second estimators (8) can be used to calculate $\hat{x}_{R,j}$ and $\hat{\beta}_j$, (e.g., in place of the MLE), if a real prior technological knowledge exists and, so, a prior interval (β_1, β_2) for β and an anticipated value $E\{x_R\}$ for x_R can be formulated.
5. Then, we can calculate the averaged robust estimates $\hat{x}_{R,0}$ and $\hat{\beta}_0$ over the all M "resamples".

Phase 2 - The empirical sampling distribution

1. We replace the unknown Weibull parameters with the estimates $\hat{x}_{R,0}$ and $\hat{\beta}_0$. At the same time, these values are used to update the prior interval (β_1, β_2) for β and the anticipated value $E\{x_R\}$ for x_R needed to calculate parameter estimates from the future small samples to be collected.
2. We generate a sufficiently large time B (i.e., 10000) of "parametric bootstrap samples" $\underline{x}^*$ of size n from the estimated Weibull distribution. Using the first estimator (8), we obtain B estimates $\hat{x}^*_{R,i}$ $(i = 1,...,B)$.
3. The frequency distribution of these estimates represents an empirical sampling distribution of the Weibull percentile estimator (8).

We must note that this distribution is conditioned to the above value $\hat{\beta}_0$ assumed for β. We used this robust $\hat{\beta}_0$ estimate (based on $M = 1000$ resamples) since the value of the shape parameter can be considered constant, being closely linked to the unchanged mechanism of failure. Technological knowledge and engineers' experience proved this assumption (Steiner and MacKay, 2001), (Zhang and Chen, 2004).

Phase 3 - The estimation of statistical control limits

The Lower Control Limit (LCL) is the value corresponding to the smallest ordered $\hat{x}^*_{R,i}$ such that $(\alpha/2) \times B$ values are below it, where α is the fixed false alarm risk.

The Upper Control Limit (UCL) is the value corresponding to the smallest ordered $\hat{x}^*_{R,i}$ such that $(\alpha/2) \times B$ values are above it.

The Center Line (CL) is the value corresponding to the median of the ordered $\hat{x}^*_{R,i}$.

In the special case we want to set a Shewhart-type control chart, we can set $\alpha = 0.0027$ and we obtain the corresponding needed control limits.

Once the control limits are obtained, the chart can be operatively used to implement the control of the Weibull percentile x_R, using the first estimator (8) at each sampling stage.

4 PERFORMANCE ANALYSIS

In this Section we report a Monte Carlo study carried out to investigate the performance of the proposed chart. As suggested in (Nichols and Padgett, 2005), the in-control Average Run Length (ARL) was calculated by generating $k = 20$ samples of size $n = 5$ from a Weibull distribution with shape parameter β_0 and percentile $x_{R,0}$. In order to comparatively evaluate the performance, we set a Shewhart-type control chart based on a theoretical false alarm risk $\alpha = 0.0027$. The control limits are calculated following the all operative steps of the procedure described in Section 3.1. Once the chart is ready to be employed, further samples are simulated from the same Weibull distribution and the percentile x_R is estimated until a point falls outside of the control limits. The in-control run length can be measured as the number of samples extracted up to and including the first out-of control signal. We repeat the whole simulation obtaining 500 replications of the run length. The in-control ARL can be computed averaging over the all 500 replicated run lengths.

In Table I we report the results of this simulation study. We considered two different lower percentiles $x_{0.90}$ and $x_{0.99}$ corresponding to the high reliability levels $R = 0.90$ and $R = 0.99$, respectively.

The chart works reliably: when the process is in-control the ARL value is close to the theoretical value 370 corresponding to $\alpha = 0.0027$.

Table I In-control ARL measures $(\alpha = 0.0027,\ n = 5)$

In-control Weibull parameters		Percentile of interest	
β_0	$x_{R,0}$	$x_{0.90}$	$x_{0.99}$
2	0.10	**383.2**	**360.5**
3	0.21	**388.2**	**379.7**

Similarly we computed the out-of-control ARLs by simulating some specific percentile shifts to be detected as critical out-of-control conditions, using the same values studied in (Nichols and Padgett, 2005). In Table II we report the results of this simulation study.

As we can see the responsiveness of the chart is good and the out-of-control ARL values are competitive with those presented in (Nichols and Padgett, 2005), being the advantage increasing as sample size of the available samples decreases.

Table II Out-of-control ARL measures $(\alpha = 0.0027,\ n = 5)$

In-control Weibull parameters		Out-of-control Weibull parameters		Percentile of interest	
β_0	$x_{R,0}$	β_1	$x_{R,1}$	$x_{0.90}$	$x_{0.99}$
1.5	0.046	1	0.01	**27.6**	**25.7**
3	0.21	2	0.10	**31.2**	**26.8**

5 CONCLUDING REMARKS

This work has shown how to effectively combine the Bayesian approach and the bootstrap techniques in order to set up a reliable control chart for Weibull percentiles.

Thanks to the bootstrap methods used, the proposed procedure enables one to estimate robust control limits and to easily implement a Shewhart-type control chart when the underlying distribution is not-Normal and the sample size is small. In the examined cases, the chart is competitive to the available alternative charts. Moreover, thanks to the Bayesian nature of the employed estimators the chart turns out to be particularly effective in critical situations when very small samples (also individuals) have to be processed

On the basis of some preliminary simulation studies the chart seems to work satisfactorily. Obviously, further and deeper research about its statistical properties is needed.

REFERENCES

Bajgier, S.M. (1992), "The use of bootstrapping to construct limits on control charts", *Proc. Decision Science Institute*, San Diego, CA, 1611–1613

Canavos, G.C., Tsokos, C.P., (1972), "Bayesian Concepts for the Estimation of Reliability in the Weibull Life-Testing Model", *International Statistical Review*, 40(2), 153 160

Canavos, G.C., Tsokos, C.P., (1973), "Bayesian Estimation of Life Parameters in the Weibull Distribution", *Operations Research*, 21(3), 755-763

Carlin, B.P., Gelfand, A.E. (1991), "A Sample Reuse Method for Accurate Parametric Empirical Bayes Confidence Intervals", *Journal of the Royal Statistical Society. Series B*, 53(1), 189-200

Erto, P. (1982), "New Practical Bayes estimators for the 2-parameters Weibull distributions", *IEEE Transactions on Reliability*, 31(2), 194-197

Erto P., (2005), "Integration of technological and statistical knowledge for reliability control", *Proc. 55th Session of International Statistical Institute*, Sydney, Australia

Erto, P., Giorgio, M. (1996), "Modified Practical Bayes Estimators", *IEEE Transactions on Reliability*, 45(1), 132-137

Erto, P., Lanzotti, A. (1990), "A Bayesian approach to the statistical control of the reliability of a flexible manufacturing system" (in Italian), *Proc. of the Annual Conf., AIRO '90, Models and methods for decision support*, Sorrento, Italy, 31-45

Erto, P., Pallotta, G. (2006), "A Bayesian Cumulative Memory Control Chart for Weibull Processes", *Proc. 9th QMOD Conference*, Liverpool, UK, 300-310

Erto, P., Pallotta, G. (2007), "A new control chart for Weibull technological processes", *Quality Technology and Quantitative Management*, 4(4), 553-567

Erto, P., Rapone, M. (1984), "Non-informative and Practical Bayesian confidence bounds for reliable life in the Weibull model", *Reliability Engineering*, 7, 181-191

Erto, P., Pallotta, G. and Park, S.H. (2008), "An Example of Data Technology Product: a Control Chart for Weibull Processes", *International Statistical Review* (available at http://www.blackwell-synergy.com/doi/abs/10.1111/j.1751-5823.2008.00043.x)

Johnson, L., Kotz, S. and Balakrishnan, N. (1994), *Continuous univariate distributions*, 2nd edition, Wiley, New York

Kanji, G.K. and Arif, O.H. (2001), "Median rankit control chart for Weibull distribution", *Total Quality Management*, 12(5), 629-642

Laird, N.M., Louis, T.A. (1987), "Empirical Bayes confidence intervals based on bootstrap samples", *Journal of American Statistical Association*, 82, 739-750

Nichols, M.D., Padgett, W.J. (2005), "A Bootstrap Control Chart for Weibull Percentiles", *Quality and Reliability Engineering International* 22(2), 141-151

Padgett, W.J., Spurrier, J.D. (1990) Shewhart-type charts for percentiles of strength distributions. *Journal of Quality Technology*, 22(4), 283-288

Park, S.H. (1996), *Robust design and analysis for quality engineering*, London, Chapman & Hall.
Seppala, T., et al. (1995), "Statistical Process Control via the Subgroup Bootstrap", *Journal of Quality Technology* 27(2), 139-153
Shapiro, S.S. (1990), *How to Test Normality and Other Distributional Assumptions*, ASQC Quality Press, Milwaukee
Shore, H. (2004), "Non-normal populations in quality applications: a revisited perspective", *Quality and Reliability Engineering International*, 20, 375-382
Smith, R.L., Naylor, J.C., (1987). "A Comparison of Maximum Likelihood and Bayesian Estimators for the Three- Parameter Weibull Distribution", *Applied Statistics*, 36(3), 358-369
Steiner, S.H., MacKay R.J. (2001), "Detecting Changes in the Mean from Censored Lifetime Data", Lenz , H.J. and Wilrich, P.T., *Frontiers in Statistical Quality Control*, Physica-Verlag, 275-289
Wood, M., Kaye, M., Capon, N., (1999), "The Use of Resampling for Estimating Control Charts Limits", *Journal of the Operational Research Society* 50(6), 651-659
Xie, M., Goh, T. N. and Ranjan P. (2002), "Some effective control chart procedures for reliability monitoring", *Reliability Engineering and System Safety*, 77, 2, 143-150
Zhang, L., Chen, G. (2004), "EWMA charts for monitoring the mean of censored Weibull lifetimes", *Journal of Quality Technology*, 36(3), 321-328